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Prove Four-Color Conjecture With "Color Chain Combination Theory"

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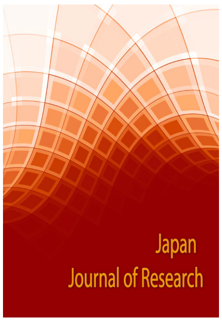

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Abstract

In the second issue of xinzhou teachers university Journal in 2004, The Perfection of Kempe's Proof was published [1]. The key points are: using the different combination theory of six color chains and the periodic H-staining program, nine configurations were constructed, which answered the question of how many Heawood configurations Kempe missed, but the discussion was not complete and accurate. In this paper, not only the generation process and detailed solution process of nine configurations are given completely, but also the completeness of nine configurations is proved by reduction to absurdity, which is an important revision and supplement of literature 1. It is different from the published method of "4-staining of dilemma configuration of four-color conjecture", but the classification of the configuration is completely consistent with the conclusion of 4- staining which can be used as a companion for readers to taste.

Introduction

Zhongfu Zhang graph theorist, published "The Trap of Mathematics" [2] in <<Nature>> in 1992, in which the mathematical induction proof of the four-color conjecture was introduced in detail (Only the core content is quoted here, and the definition is omitted):

Use d(v) to indicate the number of adjacent points of v, and use p to indicate the number of points of a planar graph.

Using mathematical induction to P, when p≤4, the conclusion is obviously valid. When it is set to p-1 (p≥5), the conclusion holds, and so does the conclusion when it is inferred to be P now.

Let d(v)=5, and the five adjacent points of V are v1, v2, v3, v4 and v5, and let {VA, VB, VC, VD} be a color division after G removes point V, and only consider the following three situations..

Case 1: v1∈VA, v2, v3∈VB, v4∈VC, v5∈VD, (Figure 1). Consider G A, C, let v1, v4 in a color diagram of G A, C there is a (V1-v4) chain, then in G B, D, there is no (v5-v2) chain and (v5-v3) chain, otherwise (V1-V4) chain and (v5-v2) or (v5-v3) chain intersect, That is, at least one point has stained both the first color and the second color (or "A" and "D", or "C" and "B", or "C" and "D"),which is contradictory, so in G B, D, the color of the points in the color division diagram where V5 is located is exchanged (" B "for" D", "D" for "B"), so that point V5 stain color B,Therefore, when point v can staining D that is P, the conclusion holds.

Case 2: v1∈VA, v2, v5∈VB, v3∈VC, v4∈VD, (Figure 2) [Author's note: This configuration is a simple configuration, by Kempe chain method, Can succeed 4-staining,we call it Kempe configuration,briefly referred K configuration]. Consider G A, C, exist (v1-v3) chain, consider G A, D,exist (v1-v4) chain, thus in G B, D; No (v2-v4) chain exists; In G B, C, there is no (v3-v5) chain. For example, in case 1, v2 can be changed into color D and v5 into color C, so that point V can staining B that is P, the conclusion holds.

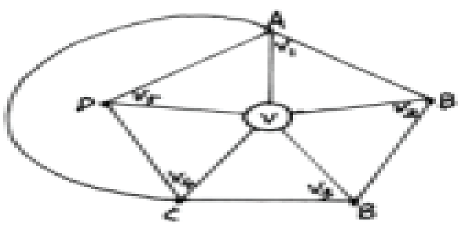

Figure 1

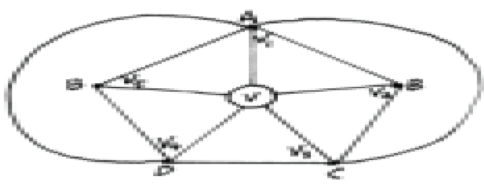

Figure 2

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Case 3: In case 2, there is another complicated situation, that is, the set (v1-v3) chain and (v1-v4) chain may intersect because they both contain color point A ,uses Heawood counter example to point out the trap in the proof of mathematical induction,he did not try to solve such a trap problem.In this paper, the following detailed supplementary proof is given for the trap problem in case 3.

Firstly, the basic model of H configuration is established, as shown in Figure 3: the colors of five adjacent points of V are A1,B1, B2, C1 and D1,Abbreviated as "B-A-B" type and the "A1- C1 ring and A1- D1 ring with V as the interface" (hereinafter referred to as"A1- C1 ring and A1- D1 ring", others are similar) once intersect A2, and B1- D2,B2- C2 and C2- D2 are adjoint generating chains.All the points in the diagram are directly represented by colors A, B, C and D, and the points in the basic model are represented by a, B, C and D with lower corner codes.

Now, according to the different quantitative combinations and different cross combinations of the six color chains inherent in the four-color map, we establish the inevitable set of H configuration and its solution with the help of the "H staining program ", that is, "staining inversion along the counterclockwise direction of B-D, A-D, A-C and B-C (4 color chains)". In the following detailed illustration, the hatched ones indicate the known or generated rings, the thick black ones indicate the color chains opposite to the rings, and they are chains with two colors reversed, and B(D), A(C) and so on indicate the color change of isolated points, and those with () are used to change staining. Next, we call "two-color inversion"as" staining inversion".

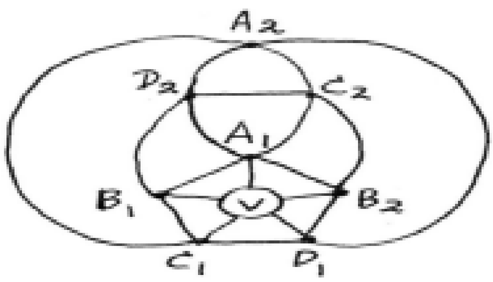

Figure 3. *Basic model of H configuration*

1. Let the B1-A2 chain and the D1-C2 chain (or the opposite B2-A2 chain) exist in Figure 3, as shown in Figure 4. The configuration of this kind of color chain combination can be staining in100version twice, Configuration 4-staining successfully.

The solution is similar to the K configuration: firstly, the B-D chain in the A1-C1 ring is staining inversion to generate a new A-D ring (if it fails, it belongs to the next configuration), and then the C-B chain outside the A-D ring is staining inversion "V"can staining C.

2. Let figure 4 have C1-D2 chain instead of B1-A2 chain, and also have D 1-C2 chain, as shown in figure 5 (this is the simplest form of Heawood counterexample configuration). After the configuration staining of this color chain combination is reversed for three times, the configuration 4 -staining successfully.

The solution procedure is as follows: B-D chain staining is reversed in A1-C1 ring to generate B-C ring; A-D chain staining reversed outside the B-C ring to generate a new A-C ring (if it fails, it belongs to the next configuration); Then the B-D chain staining reversed in the A-C ring , and V can be staining B.

3. Suppose that there are C1-D2 chains in Figure 5 and B2-A2 chains (opposite to D 1-C2 chains), as shown in Figure 6. After the configuration of this kind of color chain combination is staining reversed for 4 times, the configuration 4 -staining successfully.

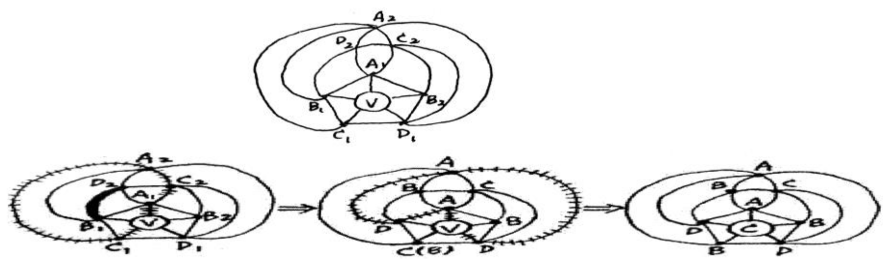

Figure 4 and its solving program

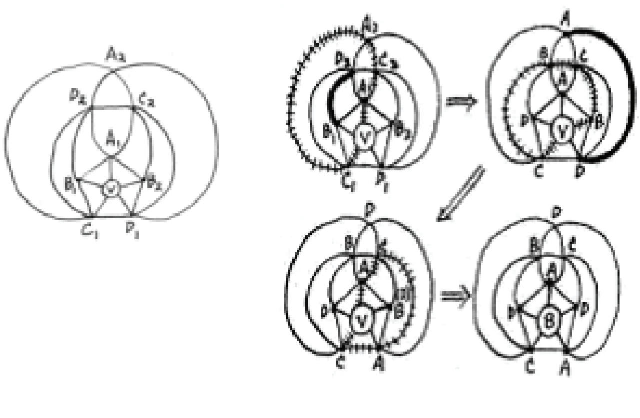

Figure 5 and its solving program

The solution procedure is as follows: B-D chain staining reversed in A1-C1 ring to generate B-C ring; The D-A chain staining reversed in B-C ring to generate a B-D ring; The A-C chain staining reversed in B-D ring, and a new B-C ring is generated (the unsuccessful situation belongs to the next configuration; The D-A chain staining reversed in B-C ring, and staining V with D color.

4. Let the B1-D2 chain in Figure 6 intersect with the A1-D1 ring, and only take one intersection to minimize the configuration. At this time, B1-A3 chain and C1-A3 chain

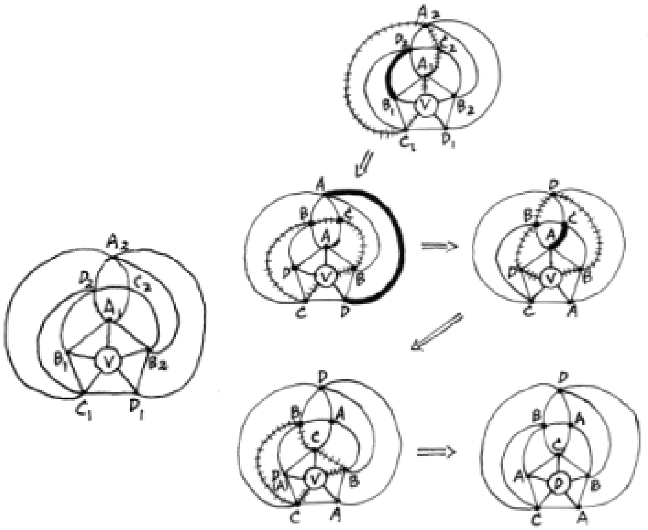

Figure 6 and its solving program

exist together, as shown in Figure 7. After the configuration of this kind of color chain combination is reversed for 5 times, the configuration 4 - staining successfully.

The solution procedure is as follows: B-D chain staining reversed in A1-C1 ring to generate B-C ring; The staining reversed of the D-A chain outside the B-C ring generate the B-D ring; The A-C chain staining reversed in B-D ring to generate A-D ring; The of the C-B chain outside the A-D ring staining reversed, and a new B-D ring is generated(if it is not born, it belongs to the next configuration); Then the staining reversed of A-C chain outside B-D ring , and V can staining A.

5. Let the C1-D2 chain in Figure 7 intersect with the A1-C1 ring. In order to minimize the configuration, only one intersection is taken, and the D color point of C1-D2

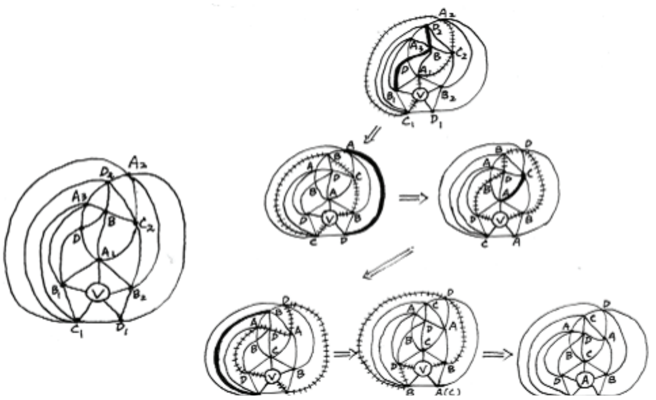

Figure 7 and its solving program

chain outside the A1-C1 ring becomes the B color point(as shown by the circle on the left in Figure 8), as shown in Figure 8. After the configuration of this kind of color chain combination is staining reversed for 6 times, the configuration 4 - staining successfully.

The solution procedure is as follows: The staining of B-D chain staining reversed in A1-C1 ring to generate B-C ring; The staining of the D-A chain outside the B-C ring staining reversed to generate the B-D ring; The staining of A-C chain in B-D ring staining reversed to generate A-D ring; The of the C-B chain outside the A-D ring staining reversed to generate the A-C ring; The of the B-D chain staining reversed outside of the A-C ring , and a new A-D ring is born (if it is not born, it belongs to the next configuration);Then the staining reversed of C-B chain in A-D ring , and V can be staining C.

6. Let the B1-D2 chain in Figure 8 intersect with the B1-A3 chain. In order to minimize the configuration, only one intersection is taken and The "A" color point in the B1-D2 chain becomes a C color point as circled in Figure 9 as shown in figure 9. After the configuration of this kind of color chain combination is staining reversed for 7 times, the configuration 4 - staining successfully.

The solution procedure is as follows: reverse the staining of B-D chain in A1-C1 ring to generate B-C ring; The staining of

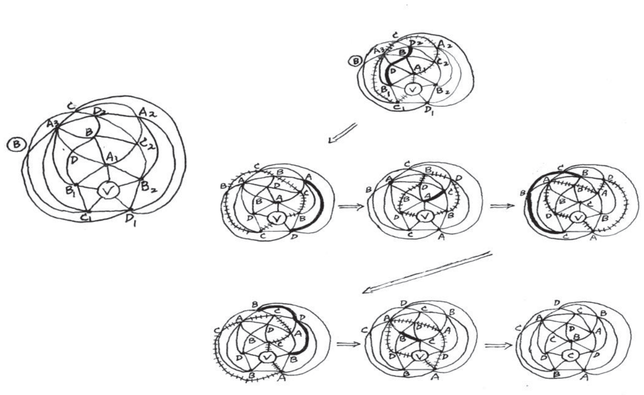

Figure 8 and its solving program

the D-A chain outside the B-C ring is reversed to generate the B-D ring; The staining of A-C chain in B-D ring is reversed to generate A-D ring; The staining of the C-B chain outside the A-D ring is reversed to generate the A-C ring; The staining of the B-D chain outside the A-C ring is reversed to generate the B-C ring; The d staining of D-A chain in B-C ring is reversed, and a new A-C ring is generated (The situation of not being born belongs to the next configuration); Then the B-D chain in the A-C ring is reversed, and V can be staining with B color.

7. Let the B1-D2 and C1-D2 chains in Figure 9 intersect in the A1-C1 ring, so as to minimize, only one intersection is taken, as shown in Figure 10. After the configuration of this kind of color chain combination is reversed for 8 times, the configuration 4 - staining successfully.

The solution procedure is as follows: reverse the staining of B-D chain in A1-C1 ring to generate B-C ring; The staining of the D-A chain outside the B-C ring is reversed to generate the B-D ring; The staining of A-C chain in B-D ring is reversed to generate A-D ring; The staining of the C-B chain outside the

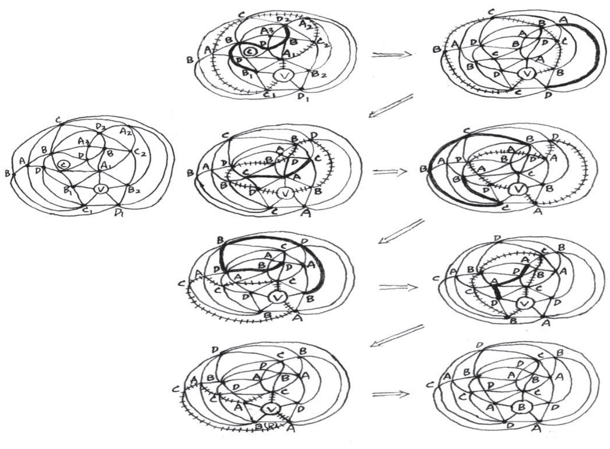

Figure 9 *and its solving program*

A-D ring is reversed to generate the A-C ring; The staining of the B-D chain outside the A-C ring is reversed to generate the B-C ring; Make the staining of the D-A chain in the B-C ring reverse to generate the B-D ring; The staining of the A-C chain outside the B-D ring is reversed, and a new B-C ring is generated (the failure situation belongs to the next configuration); Then reverse the staining of the D-A chain in the B-C ring, and you can stain V with D color.

8. Let the B2-A2 and A1-D1 rings intersect once in Figure 10, as shown in Figure 11. After the configuration of this kind of color chain combination is reversed for 9 times, the configuration 4 -staining successfully.

The solution procedure is as follows: reverse the staining of B-D chain in A1-C1 ring to generate B-C ring; The staining of the D-A chain outside the B-C ring is reversed to generate the B-D ring; The staining of A-C chain in B-D ring is reversed to generate A-D ring; The staining of the C-B chain outside the A-D ring is reversed to generate the A-C ring; The staining of the B-D chain outside the A-C ring is reversed to generate the B-C ring; Make the staining of the D-A chain in the B-C ring reverse to generate the B-D ring; The staining of A-C chain outside B-D ring is reversed to generate A-D ring; The staining of C-B chain in A-D ring is reversed, and a new B-D ring is generated. Then the staining of A-C chain in B-D ring is reversed, and V can be staining with A color.

9. In Figure 5, according to the intersection combination of Figure 8, let C1-D2 chain intersect with A1-C1 ring minimum (the so-called minimum intersection is the intersection of color chains with as few sides as possible), and D1-C2 chain intersects with A1-D2 ring minimum, and change the D color point of C1-D2 chain outside A1-C1 ring and the C color point of D1-C2 chain outside A1-D2 ring. Then the C-B chain and D-B chain outside the A1-C1 ring and the A1-D2 ring are minimally intersected, as shown in Figure 12.

If the solution method in Figure 4 to Figure 11 H staining program is followed, the Heawood configuration transformed in four periods will be generated and cannot be solved. If the H-staining program is carried out for 20 times continuously, the configuration is continuously converted clockwise for 20 times and then turned back to the initial position [5]. However, the four continuously transformed Heawood configurations have a common H- staining procedure feature, that is, they all contain A-B rings, so the following special Zhang Yudian staining procedure (Z staining procedure for short) is produced:

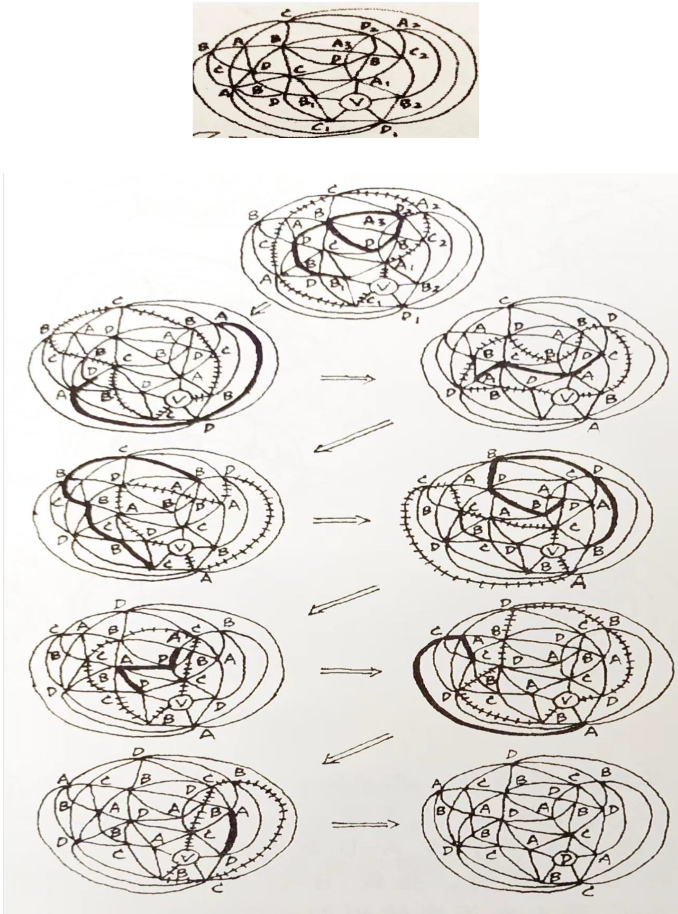

Figure 10 *and its solving program*

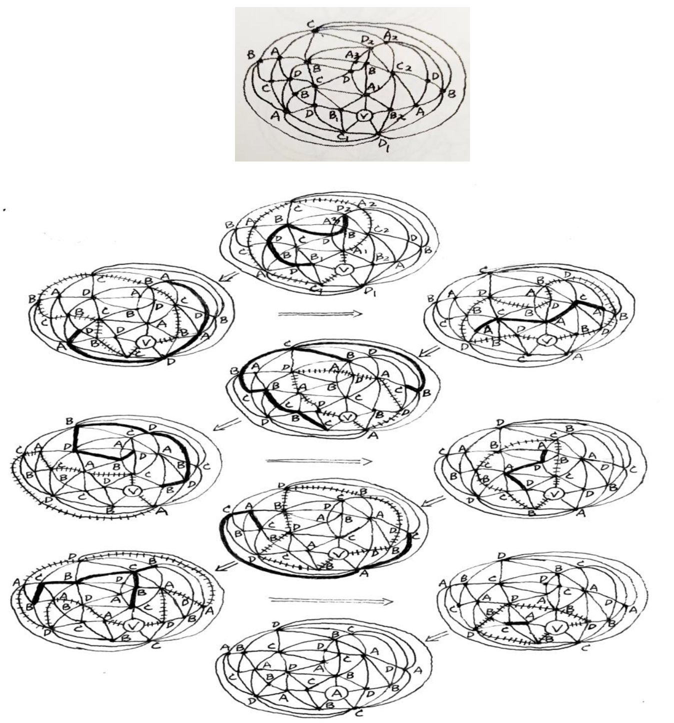

Figure 11 *and its solving program*

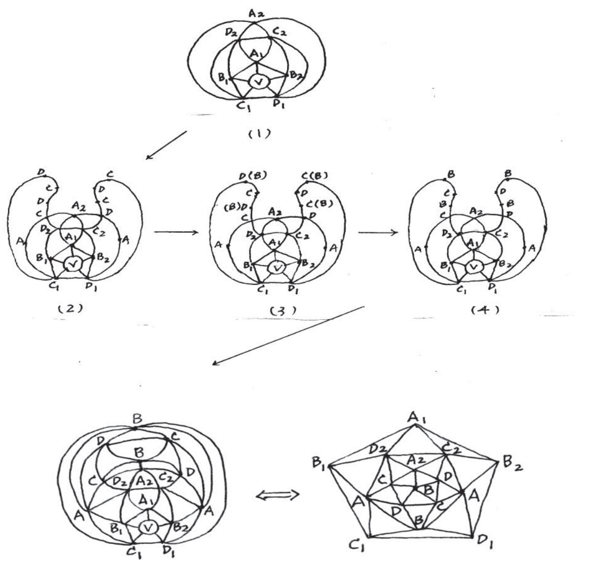

Figure 12. *The right figure is equivalent to Example 2 in Reference [4], and the left and right graphs are topological transformation graphs.*

If the graph (1) [or (3)] is known, the C and D coloring of the C-D chain outside the A-B ring is reversed to generate a new disjoint A-C, A-D (or B-C, B-D) maximal chain, similar to the case 2 (Figure 2) Then the coloring of B(D), B(C) [or A(D), A(C)] is reversed, so that the coloring number of the five vertices of the pentagon is reduced to 3. See Figure 13(1) and (3).

If the figure (2) [or (4)] is known, the C and D coloring of the C-D chain outside the A-B ring is reversed to generate a new B-C (or A-D) maximal chain, similar to the case 1 (Figure 1). Then the A(D) [or B(C)] staining on one side of the B-C (or A-D) chain is reversed, so that the coloring number of the five vertices of the pentagon is reduced to 3. See Figure13(2) and (4).

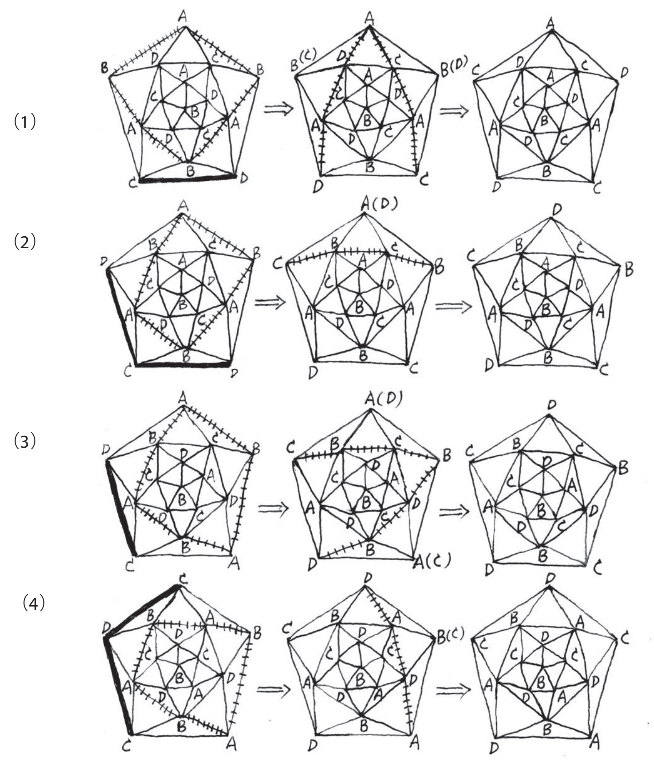

Figure 13 *and its solving program*

The following is a theoretical proof of the completeness of the inevitable set composed of Figures 4 to 13.

Reflected in Heawood's model, there are rings A1-C1 and A1-D1 which start and end at five adjacent points of V, and Color chain B1-D2, B2-C2, B1-A2 (B2-A2), C1-D2 (D1-C2), which start at five adjacent points of V and end at the edge of the intersection area of two rings A1-C1 and A 1-D1. There are four different quantity combinations of these four chains in Heawood model:

(1) B1-A2, B1-D2, B2-C2, B2-A2;

(2) B1-A2, B1-D2, B2-C2, D1-C2;

(3) C1-D2, B1-D2, B2-C2, B2-A2;

(4) C1-D2, B1-D2, B2-C2, D1-C2.

And there are 15 combinations of different positions of any two of the six color chains. There are three groups of disjoint combinations (that is, three groups of opposite color chains):

A-B and C-D, A-C and B-D, A-D and b-c;

There are 12 kinds of intersectable combinations:

A-B and A-C, A-D, B-C, B-D; A-C and A-D, B-C, C-D;

A-D and B-D, C-D; B-C and B-D, c-d; B-D and C-D.

We take the different quantitative combinations (4 kinds) and different position combinations (12 kinds of intersecting) of the above six color chains as two major variables, and get nine H configurations of different color chain combinations that conform to the H color program: Figures 4 to 6 show four different quantitative combinations, of which Figure 4 shows the combination of (1) and (2), and Figure 5 shows the combination of (3), Figure 6 shows the combination of (4);

Figure 7 to 11 shows the increasing number of cross combinations when the number of combinations in Figure 6 remains the same. Figure 11 contains 12 cross combinations.

Figure 12 embodies the special combination of symmetrical intersection and discoloration on the left and right sides of the C-D chain in Figure 5.

On the other hand, each configuration has two different results when implementing the H-staining program, either making the configuration have a solution or (failing to produce the situation belongs to the next configuration), resulting in the continuous increase of the number of staining inversions between two adjacent configurations, without gaps, that is, there are no other intermediate configurations. Therefore, it is shown that eight configurations of aperiodic cycles and one configuration of periodic cycles can and can only be obtained in the complete 9-time staining inversion.

From the above two aspects, it can be seen that different quantitative combinations and intersection combinations (theory) of color chains determine different configurations , and different configurations determine different H and Z staining procedures. There has been one-to-one correspondence between the three, which is impeccable and complete.

Let's prove the completeness of this inevitable set by reducing to absurdity.

After we have established the first eight configurations using the theory of color chain combinations and the H-staining program, is there a ninth configuration that can be 4-staining with the H-staining program? This is the question posed to us by many "four-color experts". We said there would be no. If we assume that it exists, then it must not generate A B-D ring in the penultimate diagram of the detail diagram in Figure 11, but should generate its opposite color chain A-C ring, because this can make the double B sandwich A configuration fully transformed two cycles, and its external A-C, A-D two rings still appear to intersect. In this case, the two A-colored vertices of the B-A-D-A quadrilateral containing the dotted line must be adjacent (that is, connected to the same A-C ring) so that the B-D ring is disconnected, as shown in the left figure of Figure 14.

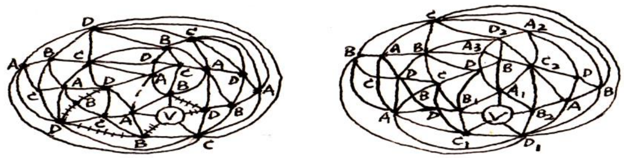

Figure 14

(The left picture shows the change of the penultimate detail of Figure 11, and the right picture shows the original Figure 11 that changes with the left picture, that is, the ninth configuration that we assume can be 4- staining.)

In this way, if you move to the left in the detail drawing, the vertices B1 and D of the A1-B1-C-D quadrilateral in the initial drawing of Figure 11 will be adjacent, rather than A1 and C, as shown in the right drawing of Figure 14. Since the B1-D2 chain is the shortest color chain in this configuration, when the H-staining procedure is applied to it, the configuration 4- staining is successfully reversed through four color changes, as shown in Figure 6. That means the ninth configuration doesn't exist

Alternatively, we can negate the starting assumption by directly examining the main effects of the new 14 right diagram on the detailed diagram of the original 11. Please see the detailed diagram of the original figure 11: In the fourth figure, the B-C chain in the second color exchange generates the A-C ring in the fifth figure, which contains the shortest chain A1-C in the initial figure 11. Otherwise, if there is no such shortest chain, there must be the shortest B1-D chain in the 11 right figure. At this time, the new A-C ring cannot be generated, and the B-D ring with the opposite ring will be generated. It shows that the original double B sandwiched A configuration of H staining program even one cycle can not be converted. Without a periodic transformation of the configuration, there is no basis for the generation of the subsequent configuration, and how to talk about the existence of the ninth configuration! In other words, the second cycle of the double-B-clip A-configuration transformation requires the B1-D chain in the initial figure 11, while its transformation of the first cycle requires the shortest chain A1-C chain in the initial figure 11, which is contradictory. So the hypothesis is not valid.

At this time, it is also reducible for the complex situation of case 3, that is, the conclusion holds when it is p.

Based on the above three situations, we know from the principle of induction that the four-color conjecture holds when d(v)=5.

Fortunately, this minimum inevitable configuration set actually includes Heawood counterexample, Holroyd- Miller counterexample (Example 2 in Ref 4), that is, the staining dilemma configuration with ten fold symmetry discovered by Errera in 1921 [6], and the three reduced obstacle configurations of Heech can also be included in it [7].

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