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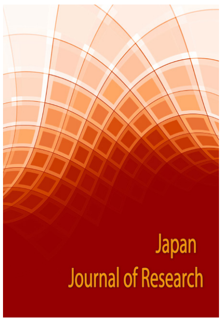

A Two-Step Longstaff Schwartz Monte Carlo Approach to Game Option Pricing

Ce Wang

Correspondence

Ce Wang

E-mail: ryan.wang.19@outlook.com

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Abstract

We proposed a two-step Longstaff Schwartz Monte Carlo (LSMC) method with two regression models fitted at each time step to price game options. Although the original LSMC can be used to price game options with an enlarged range of path in regression and a modified cashflow updating rule, we identified a drawback of such approach, which motivated us to propose our approach. We implemented numerical examples with benchmarks using binomial tree and numerical PDE, and it showed that our method produces more reliable results comparing to the original LSMC.

Introduction

Game options, also known as Israeli options, proposed by Kifer [1], is an option with characteristics of an American put option, which simultaneously offers the issuer to recall the certificate with a penalty paid to the holder. Kifer [1] showed that in the Black-Scholes market (Black and Scholes [2]) where the prices movement of the underlying asset follows

$$dS = \mu S_t dt + \sigma S_t W_t \quad (1)$$

Where S is the stock price, μ, r, σ are constant stock return, risk-free interest rate, and volatility, and W_t is a standard Brownian motion, the game option has unique non-arbitrage price.

The pricing of game option is a difficult problem. Kuhn et al. [3] proposed a simulation framework inspired by Roger [4], and also provided benchmarks via the Canadization method proposed by Carr [5]. The pricing of game options via BSDE has been studied by both Dumitrescu et al. [6], and Essaky and Hassani [7]. Wang and Hu [8] studied the pricing of game options under the Levy market. Some recent research in the pricing of game options includes, for example, Zaevski [9,10], Guo et al. [11], and Meyer [12].

The penalty paid by the issuer when recalling the option at time t, is the pay off that the holder would get if the option is exercised at time t, plus a penalty δ. Kifer [13] noted that in the discrete case, the price of a game option, denoted by $G_{i,j}$, where i and j are in the range of stock price and time, can be expressed as

$$G_{ij} = min\left(Exercise + \delta, max\left(Exercise, Holding\right)\right) \quad (2)$$

This is very useful when using numerical approaches to price game options. For example, if we use a lattice (binomial tree) or a grid (numerical PDE), the above equation can be directly used to update the node values. The idea can be also used to modify the steps in the Longstaff-Schwartz Monte Carlo method (LSMC) [14], which was originally proposed to price American put options.

The LSMC method is a simulation framework which address early-exercise properties an option has. It starts from making Monte Carlo simulation of the underlying stock, and targets to calculate cash flows corresponding to each stock path, under the mechanics of American put options. The option price is estimated by the average of the cash flows discounted to $t = 0$. The cash flows are calculated by a backward induction from the maturity date. In order to make early exercise decisions, at each time step we fit a regression model to predict the holding value, which is compared to the exercise value to decide if the option should be exercised early.

The existing literature on applying the LSMC method on game options, however, concentrates on experimental results with callable putable bonds, and callable convertible bonds, as two cases of game options. For instance, see Luo and Zhang [15], Lee et al. [16], and Kind and Wilde [17]. There have been limited technical details of the implementation process provided in these papers. We noticed that in the case of pricing stock game options, an extra regression model is needed to be built on each backward

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induction step. This extra step has never been discussed in the existing literature. Without implementing this step, the option price estimated by the LSMC method would significantly deviate from option price estimated by deterministic methods, such as binomial tree, or numerical PDEs. We named the framework of Longstaff-Schwartz Monte Carlo with this extra regression step as two-step LSMC method. In the future chapters, we will discuss the details of this method, and also leveraged (2) to implement the binomial tree and numerical PDE approaches, to serve as benchmarks.

Methodology

In this section, we will introduce our 2-step LSMC method for pricing game options, and briefly discuss why it is better than directly applying the original LSMC method when pricing game options. The performance improvement of our method is presented in numerical examples in section 3. We then explain how to modify the binomial tree and numerical PDE approach to price game options, as they are benchmarks in the numerical examples in section 3.

Two-Step Longstaff - Schwartz Monte Carlo for Game Option

Longstaff and Schwartz proposed a Monte Carlo framework incorporating linear regression to predict the holding value of an American put options, which is known as the LSMC. The LSMC method simulates n paths of stock price under the Black-Scholes model, and results n cashflows. The option price is thus the discounted average of the cash- flows. At each time step, a linear regression model is built to use stock prices to predict discounted future cashflow, and the predictions are used to make early exercise decision. When American put options are priced, we can clearly identify the paths which are relevant (i.e. the in-the-money paths), and only include those paths in the regression. However, when game options are priced, we have to include all paths, since an out-of-the-money path can have a discounted expected holding value larger than the penalty, and thus relevant to the issuer's option to recall the contract. This is a draw back and is the motivation to our method.

Although all paths are potentially relevant since the issuer has the option to exercise with penalty, in reality, many of the out-of-the-money paths are eventually not relevant at each t, i.e. their discounted expected holding values are smaller than the penalty. Containing these paths in the regression step will impair the regression models fitted at each time step. We propose the two- step LSMC method to address the above issue. In our method, we fit two linear regression models on each backward induction step. One model is fit using the in-the-money paths to estimate the corresponding holding value, to derive the exercise behaviour of both the holder and the issuer. Another model is fit using the out-of-the-money paths to estimate the corresponding holding value, to derive the exercise behaviour of the issuer. The rest part of the method is the same with the original LSMC method. This approach helps us to avoid building one regression model with all paths, which improves the performance of the second regression model. Below is a step-to-step explaination of the two-step LSMC method:

- Simulate N stock paths under the risk neutral measure of Black-Scholes model from $t \in \{0, ..., T\}$
- Initialize the corresponding cash flow of the N paths as $CF_i = max(K - S_{iT}, 0)$ where $i \in \{1, ..., N\}$
- Identify the set $N_{out\text{-}of\text{-}the\text{-}money}$ where the paths are out-of-the-money at $t = T - 1$
- Fit a linear regression model $\hat{f}_1$ where the y variable is the discounted cash flow from paths in $N_{\text{out-of-the-money}}$, and the x variables are the stock prices from paths in $N_{\text{out-of-the-money}}$, transformed by selected basis functions
- Use $\hat{f}_1$ to predict the holding value on the out-of-the-money paths, denoted as Di
- Identify the set $N_{out\text{-}of\text{-}sample\text{-}relevant}$ where the holding value is higher than the penalty
- Update CFi where $i \in N_{out\text{-}of\text{-}sample\text{-}relevant}$ as $CF_i = \delta$
- Identify the set $N_{in\text{-}the\text{-}money}$ where the paths are in-the-money at $t = T - 1$
- Fit a linear regression model $\hat{f}_2$ where the y variable is the discount cash flow from paths in $N_{in\text{-}the\text{-}money}$, and the x variables are the stock prices from paths in $N_{in\text{-}the\text{-}money}$
- Use $\hat{f}_2$ to predict the holding value of paths in $N_{in\text{-}the\text{-}money}$, denoted as C_i
- Identify the paths $N_{Exercise,\ \text{Issuer}}$ where the issuer will exercise, i.e. paths where $C_i > K - S_{i,T-1} + \delta$, and the paths $N_{Exercise,\ \text{Holder}}$ where the holder will exercise, i.e. paths where $C_i < K - S_{i,T-1}$
- Update CF_i where $i \in N_{Exercise,\ \text{Holder}}$ as $CF_i = K - S_{i,T-1}$
- Update CF_i where $i \in N_{Exercise,\ \text{Issuer}}$ as $CF_i = K - S_{i,T-1} + \delta$
- Update CF_i were i / $N_{\text{out-of-sample-relevant}} \cup N_{\text{Exercise, Holder}} \cup N_{\text{Exercise, Issuer}}$ as $CF_i = e^{-r\Delta t} CF_i$
- Repeat the above until we obtain CF_i when $t = 1$
- Compute the average of CF_i and multiply by the discount factor $e^{-r\Delta t}$ as the price of the Game option at $t = 0$.

Benchmark Numerical Methods

Cox-Ross-Rubinsten Binomial Tree

In the binomial tree approach of option pricing, a time interval $t\ [0, T]$ is discretized to N steps, and the price of the option and the underlying asset are discretized to $n + 1$ nodes at the nth step. The option prices at time T for all stock price nodes can be decided by the terminal option payoff, and the option prices at $t = 0$ can thus be determined by backward induction. Denoting Si,j as the stock price on node j at time $t = i$. The stock price movement from $t = i$ to $t = i + 1$ are specified by magnitude u and d, i.e. $S_{i+1,j} = uS_{i,j}$, and $S_{i+1,j+1} = dS_{i,j}$. u and d need to be chosen so that the limit of the stock tree follows the Black-Scholes model. We also need the risk neutral probability p of the stock moving upwards in order to construct the tree.

There are various ways to specify a binomial tree [18]. We implemented the Cox-Ross-Rubinsten tree [19], which has the following specification:

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta T}} \\
u &= e^{-\sigma\sqrt{\Delta T}} \\
p &= \frac{e^{-\sigma\sqrt{\Delta T}} - d}{u - d}
\end{aligned}
\qquad (3)
$$

Denoting $G_{T,j}$ as the option price at node j at $t = T$, then $G_{T,j} = max(K - S_{T,j}, 0)$. For time steps before T, the Game option value at a node is the minimum between the exercise value plus the penalty, and the option value of a corresponding American put option, i.e.

$$G_{t,j} = \min(\text{Exercise} + \text{Penalty}, \max(\text{Holding}, \text{Exercise}))$$
$$= \min(\max(K . S_{t,j}, 0) + + \delta, max(e^{-r\Delta T}(pG_{t+1,j} + (1-p)G_{t+1,j+1}), max(K - S_{t,j}, 0)) \quad (4)$$

Numerical PDE

In Black and Scholes [8], the following PDE known as the Black-Scholes PDE for the price of an option V is derived:

$$V_t + \frac{1}{2}\sigma^2 SV_{SS} + rSV_S - rV = 0 \qquad (5)$$

With the terminal condition V_T equals the option payoff, depends on the type of the option. In order to make the PDE to be well-posed, we propose the following boundary conditions: $V(S = 0, t) = K$, and $lim_{S\to\infty} = V(S, t) = 0$. We apply the Crank-Nicolson method to solve the Black-Scholes PDE numerically.

When solving the PDE using the finite difference method on a grid where time is discretized with step length Δt, and stock price is discretized with step length ΔS, and denoting $V_{i,j}$ as the solution of the PDE when t is on the ith grid and stock price is on the jth grid, the Crank-Nicolson [20] method makes the following approximations on the first and second order derivatives in the PDE:

$$\frac{\partial V_{i-\frac{1}{2},j}}{\partial t} = \frac{Vi,j - Vi-1,j}{\Delta t}$$
$$\frac{\partial V_{i-\frac{1}{2},j}}{\partial S} = \frac{1}{2}[\frac{V_{i-1,j+1} - V_{i-1,j-1}}{2\Delta S} + \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S}] \qquad (6)$$
$$\frac{\partial V_{i-\frac{1}{2},j}}{\partial S^2} = \frac{1}{2}[\frac{V_{i-1,j+1} - 2V_{i-1,j} + V_{i-1,j-1}}{\Delta S^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta S^2}]$$

Plugging in the above equations into the PDE results in following:

$$-\frac{\Delta t}{4}(\sigma^2 j^2 - rj)V_{i-1,j-1} + (1 + \frac{\Delta t}{2}(\sigma^2 j^2 + r))V_{i-1,j} - \frac{\Delta t}{2}(\sigma^2 j^2 + rj)V_{i-1,j+1}$$
$$= \frac{\Delta t}{4}(\sigma^2 j^2 - rj)V_{i,j-1} + (1 - \frac{\Delta t}{2}(\sigma^2 j^2 + r))V_{i,j} + \frac{\Delta t}{2}(\sigma^2 j^2 + rj)V_{i,j+1}$$

Suppose the stock price is divided into grid points $0, \ldots, N_{grid}$. Then V is specified on $i = 0$ and $i = N_{grid}$ by the boundary condition, and V on $i\ 1, \ldots, N_{grid-1}$ can be iteratively solved by the terminal condition. In particular, when Game option is priced, we have:

$$G_{t,j} = min(Exercise + Penalty, max(Holding, Exercise)$$
$$= min(max(K - S_{i,j}, 0) + \delta, max(V_{i,j}, max(K - S_{i,j}, 0)))$$

Numerical Examples

In this sections, we present some numerical examples of using the LSMC, two-step LSMC, Cox-Ross- Robinsten binomial tree, and the numerical PDE to price game options. The purpose is to examine if the performance of the two-step LSMC is better from the LSMC without the second regression step, and use the binomial tree and numerical PDE approaches as benchmarks.

$\delta = 10, S_0 = 100, r = 0.03, T = 1$				
Other Parameters	LSMC	Two-Step LSMC	Binomial Tree	Numerical PDE
σ = 0.2, K = 100	6.52313	6.75024	6.74290	6.74006
σ = 0.2, K = 110	12.55097	12.73814	12.72614	12.72369
σ = 0.2, K = 90	2.67519	2.86421	2.86320	2.86109
σ = 0.25, K = 100	8.37909	8.65983	8.67473	8.67252
σ = 0.25, K = 110	14.36542	14.57332	14.57888	14.57688
σ = 0.25, K = 90	4.20261	4.44843	4.42916	4.42726
σ = 0.15, K = 100	4.63753	4.82596	4.82058	4.81675
σ = 0.15, K = 110	11.91566	11.02098	11.04919	11.04612
σ = 0.15, K = 90	1.35713	1.46590	1.46258	1.46014

$\delta = 5, S_0 = 100, r = 0.03, T = 1$				
Other Parameters	LSMC	Two-Step LSMC	Binomial Tree	Numerical PDE
σ = 0.2, K = 100	4.63699	4.81503	4.82058	4.81675
σ = 0.2, K = 110	7.42106	7.56288	7.56807	7.56345
σ = 0.2, K = 90	2.65930	2.79722	2.80641	2.80303
σ = 0.25, K = 100	2.78096	2.91488	2.92587	2.92006
σ = 0.25, K = 110	5.69741	5.80382	5.80730	5.79815
σ = 0.25, K = 90	1.14396	1.23022	1.22526	1.22093
σ = 0.15, K = 100	3.73737	3.87501	3.86775	3.86313
σ = 0.15, K = 110	6.52621	6.65701	6.65279	6.64865
σ = 0.15, K = 90	1.89164	1.99751	1.98590	1.98217

From the above examples, it is clear that our two benchmarks via Cox-Ross- Robinsten binomial tree and numerical PDE reconciles well, and can be considered as the 'true value' of game options. Results from our two-step LSMC method are significantly closer to the benchmarks comparing to the LSMC method without the extra regression step that we propose. Therefore, we are confident to conclude that the extra regression step we propose improves the performance of the LSMC method for pricing game options, and produces trustworthy results.

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